Journal of Applied Mathematics and Computing
https://doi.org/10.1007/s12190-020-01479-5

ORIGINAL RESEARCH

$\mathbb{F}_q$-Linear skew cyclic codes over $\mathbb{F}_{q^2}$ and their applications of quantum codes construction

Yun Gao[1] · Jian Gao[2,3,4] · Shilin Yang[1] · Fang-Wei Fu[5]

Received: 8 November 2020 / Revised: 9 December 2020 / Accepted: 20 December 2020

Abstract
In this paper, we study the structure of $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$. Some good $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ are constructed. Moreover, as an application, some good quantum codes are obtained by $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$.

Keywords $\mathbb{F}_q$-Linear skew cyclic codes over $\mathbb{F}_{q^2}$ · Trace-alternating form · Quantum codes

Mathematics Subject Classification 94B05 · 94B15 · 11T71

1 Introduction

Cyclic codes form an important class of linear codes due to their good algebraic structures in coding theory and decoding theory. Skew cyclic codes are generalizations of

✉ Yun Gao
gaoyun2014@126.com

Jian Gao
dezhougaojian@163.com

Shilin Yang
slyang@bjut.edu.cn

Fang-Wei Fu
fwfu@nankai.edu.cn

1 College of Mathematics, Faculty of Science, Beijing University of Technology, Beijing 100124, China

2 School of Science, Shandong University of Technology, Zibo 255000, China

3 Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, China

4 School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, China

5 Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China

Published online: 29 March 2021

cyclic codes. Boucher et al. [4] showed that skew cyclic codes allowed to systematically search for codes with good properties and some examples of codes which improved the previously best known linear codes were obtained. Siap et al. [37] gave the structure of skew cyclic codes of arbitrary length. Recently, Ashraf and Mohammad [3] studied the structure and the idempotent generators of skew cyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, where $u^2 = u$, $v^2 = v$ and $uv = vu = 0$. Bag and Upadhyay [7] introduced the structures of skew cyclic and skew constacyclic codes over the ring $\mathbb{F}_p + u_1\mathbb{F}_p + \cdots + u_{2m}\mathbb{F}_p$. Furthermore, Dertli et al. [11], Gursoy et al. [25] and Gao et al. [17] studied skew cyclic and skew constacyclic codes over finite rings, respectively.

Quantum error-correcting codes are used in quantum communication and quantum computation to protect quantum information from errors due to the decoherence and other quantum noise. Quantum error-correcting codes provides an efficient way to overcome decoherence. Quantum error-correcting codes were first discovered by Shor [39]. Later, Calderbank et al. [8] introduced the CSS construction for constructing quantum codes from widely well-known classical error-correcting codes. Shortly afterwards, the construction of quantum error-correcting codes from codes over finite fields and finite rings has developed rapidly, such as [1,2,6,8–10,12–16,18–21,23,24,26–28,30–33,35,36,38,40].

Additive skew cyclic codes over finite fields could be used to construct quantum error-correcting codes. In 2011, Ezerman et al. [14] gave a method to construct additive asymmetric quantum codes from additive skew cyclic codes over $\mathbb{F}_4$. Recently, Aydin and Abualrub [1] studied the structure of additive skew cyclic codes over the quaternary field $\mathbb{F}_4$. And many best known and optimal quantum codes were obtained. Recently, we gave a family of cyclic $\mathbb{F}_q$-linear $\mathbb{F}_{q^t}$-codes of length n, and constructed 60 optimal cyclic $\mathbb{F}_q$-linear $\mathbb{F}_{q^2}$-codes [22], where n was a positive integer coprime to q. In this paper, we will study the structure of $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ and give a method to construct optimal and new quantum codes from these skew cyclic codes.

The paper is organized as follows. In Sect. 2, we sketch some basic results needed in this paper. Section 3 studies the structure of $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$. Some good $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ are constructed. In Sect. 4, we obtain some optimal and new quantum codes from $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$.

2 Preliminaries

Let $\mathbb{F}_{q^2}$ be a finite field of cardinality q^2, where $q = p^m$, p is a prime and m is a positive integer. Let ξ be a primitive element of $\mathbb{F}_{q^2}$ with $\mathrm{ord}(\xi) = q^2 - 1$. Then $\mathbb{F}_{q^2} = \{0, \xi, \xi^2, \ldots, \xi^{q^2-1}\}$. Consider the map $\theta : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$; $\alpha \mapsto \alpha^q$ for any $\alpha \in \mathbb{F}_{q^2}$. It is known that θ is the Frobenius automorphism of $\mathbb{F}_{q^2}$ with $|\langle\theta\rangle| = 2$. Similarly to [1, Definition 1], we have the following definition.

Definition 1 Let $\mathbb{F}_{q^2}$ be a finite field with q^2 elements and θ be the Frobenius automorphism of $\mathbb{F}_{q^2}$ with $|\langle\theta\rangle| = 2$. Let C be a subset of $\mathbb{F}_{q^2}^n$. Then C is called an $\mathbb{F}_q$-linear skew cyclic code of length n if

(i) C is an additive subgroup of $(\mathbb{F}_{q^2}^n, +)$;

(ii) If $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then $w \cdot c = (wc_0, wc_1, \ldots, wc_{n-1}) \in C$ for any $w \in \mathbb{F}_q$;
(iii) C is closed under the θ-cyclic shift, i.e., if $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ then

$$\theta(c) = (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C.$$

For any positive integer n, we consider the skew polynomial set

$$\mathbb{F}_{q^2}[x, \theta] = \{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} | \ a_i \in \mathbb{F}_{q^2}, 0 \leq i \leq n-1\}.$$

The skew multiplication denoted by $*$ in skew polynomial set $\mathbb{F}_{q^2}[x, \theta]$ is defined by the basic rule

$$(\alpha x^i) * (\beta x^j) = \alpha\theta^i(\beta)x^{i+j},$$

$0 \leq i, j \leq n-1$, $\alpha, \beta \in \mathbb{F}_{q^2}$ and $\theta^i(\beta) = \beta^{q^i}$. It is not difficult to verify that the multiplication $*$ is not commutative. By [4] and [37], we have that the skew polynomial set $\mathbb{F}_{q^2}[x, \theta]$ with respect to the usual addition of polynomials and multiplication defined above forms a non-commutative ring called the skew polynomial ring.

Let $R_n = \mathbb{F}_{q^2}[x, \theta]/\langle x^n - 1\rangle$. We identify each element $c = (c_0, c_1, \ldots, c_{n-1})$ of $\mathbb{F}_{q^2}^n$ with the polynomial $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in R_n$. It is not difficult to verify that the Frobenius automorphism θ on $\mathbb{F}_{q^2}^n$ is corresponding to the operation on R_n by multiplying x, that is

$$x * c(x) = \theta(c_{n-1}) + \theta(c_0)x + \theta(c_1)x^2 + \cdots + \theta(c_{n-2})x^{n-1}.$$

Let $f(x) + \langle x^n - 1\rangle$ be an element in the set R_n, and let $r(x) \in \mathbb{F}_{q^2}[x, \theta]$. Define multiplication from left as:

$$r(x) * (f(x) + \langle x^n - 1\rangle) = r(x) * f(x) + \langle x^n - 1\rangle.$$

According to [37], we have that the multiplication is well-defined for any positive integer n and R_n is a left $\mathbb{F}_{q^2}[x, \theta]$-module.

3 $\mathbb{F}_q$-Linear skew cyclic codes over $\mathbb{F}_{q^2}$

In this section, we study the structure of $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$. With the discussion above, we give the polynomial definition of $\mathbb{F}_q$-linear skew cyclic codes of arbitrary length n over $\mathbb{F}_{q^2}$ as follows.

Definition 2 Let C be a subset of R_n. C is called an $\mathbb{F}_q$-linear skew cyclic code if the following three conditions hold.

(i) C is a subgroup of R_n;

(ii) $w \cdot c(x) = wc_0 + wc_1x + \cdots + wc_{n-1}x^{n-1} \in C$ for any $c(x) \in C$ and any $w \in \mathbb{F}_q$;
(iii) If $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in C$, then

$$x * c(x) = \theta(c_{n-1}) + \theta(c_0)x + \theta(c_1)x^2 + \cdots + \theta(c_{n-2})x^{n-1} \in C.$$

Lemma 1 *A code C in R_n is an $\mathbb{F}_q$-linear skew cyclic code of length n if and only if C is a left $\mathbb{F}_q[x]/\langle x^n - 1\rangle$-submodule of R_n.*

Proof Suppose that C is an $\mathbb{F}_q$-linear skew cyclic code in R_n. According to Definition 2, we have that C is a subgroup of R_n and C is $\mathbb{F}_q$-linear. Furthermore, for any codeword $c(x) \in C$, by the definition of $\mathbb{F}_q$-linear skew cyclic codes, we have that $x^i * c(x) \in C$ for $0 \leq i \leq n-1$. By Definitions 1 and 2, we obtain that $f(x) * c(x) \in C$ for any $f(x) \in \mathbb{F}_q[x]$. Therefore, C is a left $\mathbb{F}_q[x]/\langle x^n - 1\rangle$-submodule of R_n.

Conversely, suppose C is a left $\mathbb{F}_q[x]/\langle x^n - 1\rangle$-submodule of R_n. Then we have that $f(x) * c(x) \in C$ for any codeword $c(x) \in C$ and $f(x) \in \mathbb{F}_q[x]$. Furthermore, for any $a(x),\ b(x) \in C$, we have $a(x) + b(x) \in C$. This means that C is a subgroup of R_n and C is $\mathbb{F}_q$-linear. Hence, C is an $\mathbb{F}_q$-linear skew cyclic code in R_n. □

Lemma 2 *[29, Theorem 2.25] Let F be a finite extension of $K = \mathbb{F}_q$. Then for $\alpha \in F$ we have $Tr_{F/K}(\alpha) = 0$ if and only if $\alpha = \beta^q - \beta$ for some $\beta \in F$.*

According to the definitions and lemmas given above, we get the structure of $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ in the following theorem.

Theorem 1 *The code C is an $\mathbb{F}_q$-linear skew cyclic code of length n over $\mathbb{F}_{q^2}$ if and only if C is a left $\mathbb{F}_q[x]/\langle x^n - 1\rangle$-submodule of R_n in the form*

$$C = \langle \xi g(x) + p(x), k(x)\rangle,$$

where $g(x) \in \mathbb{F}_q[x]/\langle x^n - 1\rangle$, $g(x) | (x^n - 1) \bmod q$, $p(x) \in \ker(\varphi)$ and $k(x) = r^q(x) - r(x) \in C$ for some $r(x) \in \mathbb{F}_{q^2}[x]/\langle x^n - 1\rangle$.

Proof Suppose that C is an $\mathbb{F}_q$-linear skew cyclic code of length n over $\mathbb{F}_{q^2}$. We define the map

$$\varphi : C \to \mathbb{F}_q[x]/\langle x^n - 1\rangle$$

by $\varphi(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) \mapsto (c_0 + \theta(c_0)) + (c_1 + \theta(c_1))x + \cdots + (c_{n-1} + \theta(c_{n-1}))x^{n-1}$ for any $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in C$ and $c_i \in \mathbb{F}_{q^2}$, $0 \leq i \leq n-1$. It is evident that φ is the trace map from $\mathbb{F}_{q^2}[x]$ to $\mathbb{F}_q[x]$. With the definition of the trace map φ, we have that $\varphi(z) = z + \theta(z) \in \mathbb{F}_q$ for any $z \in \mathbb{F}_{q^2}$. Furthermore, for any $c(x) \in \mathbb{F}_{q^2}[x]/\langle x^n - 1\rangle,\ d(x) \in \mathbb{F}_{q^2}[x]/\langle x^n - 1\rangle$, we have

$$\varphi(c(x) + d(x)) = \varphi(c(x)) + \varphi(d(x))$$

and

$$\begin{aligned}
&\varphi(wx^i * (c_0 + c_1x + \cdots + c_{n-1}x^{n-1})) \\
&= \varphi(w\theta^i(c_0)x^i + w\theta^i(c_1)x^{i+1} + \cdots + w\theta^i(c_{n-1})x^{n-1+i}) \\
&= (w\theta^i(c_0) + w\theta^{i+1}(c_0))x^i + (w\theta^i(c_1) + w\theta^{i+1}(c_1))x^{i+1} + \cdots \\
&\quad + (w\theta^i(c_{n-1}) + w\theta^{i+1}(c_{n-1}))x^{n-1+i} \\
&= wx^i(\theta^i(c_0) + \theta^{i+1}(c_0) + (\theta^i(c_1) + \theta^{i+1}(c_1))x + \cdots \\
&\quad + (\theta^i(c_{n-1}) + \theta^{i+1}(c_{n-1}))x^{n-1}) \\
&= wx^i * \varphi(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) \mod (x^n - 1)
\end{aligned}$$

for any $w \in \mathbb{F}_q$ and $0 \leq i \leq n-1$. This means that

$$\varphi(f(x) * c(x)) = f(x) * \varphi(c(x))$$

for any $f(x) \in \mathbb{F}_q[x]/\langle x^n - 1\rangle$. Therefore, φ is an $\mathbb{F}_q[x]/\langle x^n - 1\rangle$ module homomorphism.

By Lemma 2, it is not difficult to see that

$$\ker(\varphi) = \left\{k(x) | \exists\, r(x) \in \mathbb{F}_{q^2}[x]/\left\langle x^n - 1\right\rangle \text{ such that } k(x) = r^q(x) - r(x) \in C\right\},$$

where $r^q(x) = r_0^q + r_1^q x + \cdots + r_{n-1}^q x^{n-1}$. Let $k(x) \in \ker(\varphi)$ and $f(x) \in \mathbb{F}_q[x]/\langle x^n - 1\rangle$. Then we have

$$\varphi(f(x) * k(x)) = f(x) * \varphi(k(x)) = f(x) * 0 = 0.$$

Therefore, $\ker(\varphi)$ is an $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_{q^2}[x]/\langle x^n - 1\rangle$.

Suppose that $\varphi(c(x)) = b(x) \in \mathrm{Im}(\varphi)$ and let $f(x) \in \mathbb{F}_q[x]/\langle x^n - 1\rangle$, where $c(x) \in C$. Then

$$\varphi(f(x) * c(x)) = f(x) * \varphi(c(x)) = f(x) * b(x) \in \mathrm{Im}(\varphi).$$

Therefore, we obtain that $\mathrm{Im}(\varphi)$ is an ideal in $\mathbb{F}_q[x]/\langle x^n - 1\rangle$ and it is an $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_q[x]/\langle x^n - 1\rangle$. Thus, we have $\mathrm{Im}(\varphi) = \langle g(x)\rangle$ for some $g(x) | (x^n - 1) \bmod q$. Then, by the first module isomorphism theorem, we have that

$$C/\ker(\varphi) \cong \langle g(x)\rangle .$$

Let $\xi g(x) + p(x) \in C$ such that $\varphi(\xi g(x) + p(x)) = (\xi + \xi^q)g(x) = ag(x)$, where $p(x) \in \ker(\varphi)$ and $a \in \mathbb{F}_q \setminus \{0\}$. Since $\langle ag(x)\rangle = \langle g(x)\rangle$, we have that C is generated by two elements of the form

$$C = \langle \xi g(x) + p(x), k(x)\rangle,$$

where $k(x) = r^q(x) - r(x) \in C$ for some $r(x) \in \mathbb{F}_{q^2}[x]/\langle x^n - 1\rangle$.

Inversely, suppose C is a left $\mathbb{F}_q[x]/\langle x^n-1\rangle$-submodule of R_n given by

$$C = \langle \xi g(x) + p(x), k(x)\rangle,$$

where $g(x) \in \mathbb{F}_q[x]/\langle x^n-1\rangle$, $g(x)|(x^n-1) \bmod q$, $p(x) \in \ker(\varphi)$ and $k(x) = r^q(x) - r(x) \in C$ for some $r(x) \in \mathbb{F}_{q^2}[x]/\langle x^n-1\rangle$. Then, by Lemma 1, we have that C is an $\mathbb{F}_q$-linear skew cyclic code of length n over $\mathbb{F}_{q^2}$. □

Lemma 3 *Let $C = \langle \xi g(x) + p(x), k(x)\rangle$ be an $\mathbb{F}_q$-linear skew cyclic code of odd length n over $\mathbb{F}_{q^2}$. Then $g(x) \in C$.*

Proof Let $c(x) = \xi g(x) + p(x) \in C$, where $g(x) \in \mathbb{F}_q[x]/\langle x^n-1\rangle$, $g(x)|(x^n-1) \bmod q$ and $p(x) \in \ker(\varphi)$. Without loss of generality, we may assume that $p(x) = r_1^q(x) - r_1(x) \in C$ for some $r_1(x) \in \mathbb{F}_{q^2}[x]/\langle x^n-1\rangle$. By the definition of the skew multiplication $*$, we have that

$$x^n * c(x) = \xi^q g(x) + r_1(x) - r_1^q(x) \in C,$$

where n is odd. Hence, we have

$$\begin{aligned} x^n * c(x) + c(x) &= (\xi^q + \xi)g(x) + r_1(x) - r_1^q(x) + r_1^q(x) - r_1(x) \\ &= (\xi^q + \xi)g(x) \in C. \end{aligned}$$

Since $\xi^q + \xi \in \mathbb{F}_q \setminus \{0\}$, by Definition 2, we have that $g(x) \in C$. □

According to Theorem 1 and Lemma 3, we give some good $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ with $k(x) = 0$ which have the same parameters with the best known linear codes given in [34].

Example 1 Let $q = 2$ and $n = 8$. Then $x^8 - 1 = (x+1)^8$ over $\mathbb{F}_2$. Let $g(x) = (x+1)^3 = x^3 + x^2 + x + 1$ and $p(x) = x^6 + x^2 + 1$. Then

$$C = \langle \xi_1 g(x) + p(x)\rangle = \langle x^6 + \xi_1 x^3 + (1+\xi_1)x^2 + \xi_1 x + 1 + \xi_1\rangle$$

with generator matrix

$$G = \begin{pmatrix} 1+\xi_1 & \xi_1 & 1+\xi_1 & \xi_1 & 0 & 0 & 1 & 0 \\ 0 & (1+\xi_1)^2 & \xi_1^2 & (1+\xi_1)^2 & \xi_1^2 & 0 & 0 & 1 \\ 1 & 0 & (1+\xi_1)^4 & \xi_1^4 & (1+\xi_1)^4 & \xi_1^4 & 0 & 0 \\ 0 & 1 & 0 & (1+\xi_1)^8 & \xi_1^8 & (1+\xi_1)^8 & \xi_1^8 & 0 \\ 0 & 0 & 1 & 0 & (1+\xi_1)^{16} & \xi_1^{16} & (1+\xi_1)^{16} & \xi_1^{16} \\ \xi_1^{32} & 0 & 0 & 1 & 0 & (1+\xi_1)^{32} & \xi_1^{32} & (1+\xi_1)^{32} \\ (1+\xi_1)^{64} & \xi_1^{64} & 0 & 0 & 1 & 0 & (1+\xi_1)^{64} & \xi_1^{64} \\ \xi_1^{128} & (1+\xi_1)^{128} & \xi_1^{128} & 0 & 0 & 1 & 0 & (1+\xi_1)^{128} \end{pmatrix},$$

Table 1 Some good $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$

q	n	$g(x)$	$p(x)$	$(n, (q^2)^k, d)_{q^2}$
2	4	1	x^3+x^2+1	$(4, (2^2)^2, 3)_4$
2	6	x^3+1	x^2+x	$(6, (2^2)^2, 4)_4$
2	8	x^4+1	$x^6+x^5+x^2+x$	$(8, (2^2)^2, 6)_4$
2	8	x^3+x^2+x+1	x^6+x^2+1	$(8, (2^2)^4, 4)_4$
2	11	a_1	0	$(11, (2^2)^1, 11)_4$
2	13	a_2	a_2	$(13, (2^2)^1, 13)_4$
3	4	1	$\xi_2^2x^3+\xi_2^6x^2+\xi_2^2$	$(4, (3^2)^2, 3)_9$
3	6	x^2+2x+1	$\xi_2^2x^3+\xi_2^6x^2$	$(6, (3^2)^3, 4)_9$
3	10	a_3	0	$(10, (3^2)^1, 10)_9$
3	11	a_4	$\xi_2^2a_4$	$(11, (3^2)^1, 11)_9$

where ξ_1 is a primitive element of $\mathbb{F}_{2^2}$ with $\mathrm{ord}(\xi_1)=3$. By computer system Magma [5], we have that C is a $(8, (2^2)^4, 4)$ $\mathbb{F}_2$-linear skew cyclic code over $\mathbb{F}_4$ which has the same parameters as that of the best known linear code $[8, 4, 4]$ over $\mathbb{F}_4$ given in [34].

At the end of this example, we list some good $\mathbb{F}_q$-linear skew cyclic codes $(n, (q^2)^k, d)$ over $\mathbb{F}_4$ and $\mathbb{F}_9$ which have the same parameters with the best known linear codes given in [34] in Table 1. In Table 1 n is the length of C, k is the dimension of C and d is the minimum Hamming distance of C.

Remark 1 In Table 1,

$$a_1 = x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1,$$
$$a_2 = x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1,$$
$$a_3 = x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1,$$
$$a_4 = x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$$

and ξ_2 is a primitive element of $\mathbb{F}_{3^2}$ with $\mathrm{ord}(\xi_2)=8$.

Now we define a map $\mathcal{S}$ from $\mathbb{F}_{q^2}^n$ to $\mathbb{F}_{q^2}^{2n}$ as follows

$$\begin{aligned}\mathcal{S}: \quad \mathbb{F}_{q^2}^n &\to \mathbb{F}_{q^2}^{2n}\\ (c_0,\ldots,c_{n-1}) &\mapsto (c_0,\theta(c_0),\ldots,c_{n-1},\theta(c_{n-1})).\end{aligned}$$

Clearly, $\mathcal{S}$ is an $\mathbb{F}_q$-linear map, injective but not surjective. Let $n=2$ and $c=(\xi,\xi^q)$, where ξ is a primitive element of $\mathbb{F}_{q^2}$. Then we have

$$\mathcal{S}(c) = (\xi,\xi^q,\xi^q,\xi),$$
$$\mathcal{S}(\xi\cdot c) = \mathcal{S}(\xi^2,\xi^{q+1}) = (\xi^2,\xi^{2q},\xi^{q+1},\xi^{q+1})$$

and

$$\xi\cdot\mathcal{S}(c) = \xi\cdot(\xi,\xi^q,\xi^q,\xi) = (\xi^2,\xi^{q+1},\xi^{q+1},\xi^2).$$

Since $\mathcal{S}(\xi \cdot c) \neq \xi \cdot \mathcal{S}(c)$, we have that $\mathcal{S}$ is not an $\mathbb{F}_{q^2}$-linear map.

Let $(n, M, d)_{q^2}$ denotes a code C of length n over $\mathbb{F}_{q^2}$, where M is the cardinality of C and d is the minimum Hamming distance of C. By the definition of $\mathcal{S}$, we have the following lemma immediately.

Lemma 4 *Let C be a code of length n over $\mathbb{F}_{q^2}$ with parameters $(n, M, d)_{q^2}$. For any $c = (c_0, \ldots, c_{n-1}) \in C$, we have*

$$wt_H(\mathcal{S}(c)) = 2wt_H(c)$$

and

$$d(\mathcal{S}(C)) = 2d(C),$$

where $wt_H(c)$ denotes the Hamming weight of c and $d(C)$ denotes the minimum Hamming distance of C.

Proof According to the definition of the map $\mathcal{S}$, for any $c = (c_0, \ldots, c_{n-1}) \in C$, we have that

$$\mathcal{S}(c) = (c_0, c_0^q, \ldots, c_{n-1}, c_{n-1}^q).$$

It is not difficult to show that, $c_i^q = 0$ if and only if $c_i = 0$ for $0 \leq i \leq n-1$. So, if $c_i \neq 0$, then we have $c_i^q \neq 0$. The proof of the lemma is now complete. □

Lemma 5 *If C is an $\mathbb{F}_q$-linear skew cyclic $(n, M, d)_{q^2}$ code, then $\mathcal{S}(C)$ is an additive skew $(2n, M, 2d)_{q^2}$ 2-quasi-cyclic code.*

Proof As C is an $\mathbb{F}_q$-linear skew cyclic code over $\mathbb{F}_{q^2}$, by the definition of C, we have that $c = (c_0, \ldots, c_{n-1}) \in C$ if and only if $\theta(c) = (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C$. Applying $\mathcal{S}$ on c and $\theta(c)$, we get

$$\mathcal{S}(c) = (c_0, \theta(c_0), \ldots, c_{n-1}, \theta(c_{n-1})) \in \mathcal{S}(C)$$

if and only if

$$\mathcal{S}(\theta(c)) = (\theta(c_{n-1}), c_{n-1}, \theta(c_0), c_0, \ldots, \theta(c_{n-2}), c_{n-2}) \in \mathcal{S}(C).$$

According to Lemma 4, we have that $\mathcal{S}(C)$ is an additive skew $(2n, M, 2d)_{q^2}$ 2-quasi-cyclic code. □

4 Quantum codes from $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$

In this section, we give a method to construct quantum codes from $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$. We consider a trace-alternating form of two vectors b and c in $\mathbb{F}_{q^2}^n$ by

$$< b, c >_a = tr_{q/p}\left(\frac{b \cdot c^q - b^q \cdot c}{\xi^{2q} - \xi^2}\right),$$

where ξ is a primitive element of $\mathbb{F}_{q^2}$, $q = p^m$ and $tr_{q/p}$ is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$ defined by

$$\begin{aligned} tr_{q/p} : \mathbb{F}_q &\to \mathbb{F}_p \\ \beta &\mapsto \beta + \beta^p + \cdots + \beta^{p^m - 1}. \end{aligned}$$

It is easy to see that $< b, c >_a \in \mathbb{F}_q$. By [27], we have that the trace-alternating form is bi-additive, and it is $\mathbb{F}_p$-linear, but not $\mathbb{F}_q$-linear. Furthermore, it is alternating in the sense that $< b, b >_a = 0$ holds for all $b \in \mathbb{F}_{q^2}^n$.

Let C be an $\mathbb{F}_q$-linear skew cyclic code of length n over $\mathbb{F}_{q^2}$. Its trace-alternating dual code is defined as

$$C^{\perp_a} = \{u \in \mathbb{F}_{q^2}^n \mid < u, c >_a = 0 \text{ for all } c \in C\}.$$

Furthermore, C is said to be self-orthogonal if $C \subseteq C^{\perp_a}$, dual containing if $C^{\perp_a} \subseteq C$ and self-dual if $C = C^{\perp_a}$.

Lemma 6 *Let C be an $\mathbb{F}_q$-linear skew cyclic code over $\mathbb{F}_{q^2}$ with parameters $(n, M, d)_{q^2}$. Then*

$$\mathcal{S}(C) \subseteq \mathcal{S}(C)^{\perp_a}.$$

Proof For any $b = (b_0, b_1, \ldots, b_{n-1}) \in C$, $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, by the definition of the trace-alternating form, we have that

$$< \mathcal{S}(b), \mathcal{S}(c) >_a = tr_{q/p}\left(\sum_{i=0}^{n-1} \frac{b_i c_i^q + b_i^q c_i}{\xi^{2q} - \xi^2} - \sum_{i=0}^{n-1} \frac{b_i^q c_i + b_i c_i^q}{\xi^{2q} - \xi^2}\right) = tr_{q/p}(0) = 0.$$

This implies that for any $\mathcal{S}(b), \mathcal{S}(c) \in \mathcal{S}(C)$, we have $< \mathcal{S}(b), \mathcal{S}(c) >_a = 0$. Thus we get $\mathcal{S}(C) \subseteq \mathcal{S}(C)^{\perp_a}$. □

Let $[[n, k, d]]_q$ denotes a quantum code over $\mathbb{F}_q$ with length n, dimension k and minimum Hamming distance d. The following lemma is useful for our results.

Lemma 7 *[27, Theorem 15] An $((n, K, d))_q$ stabilizer code exists if and only if there exists an additive subcode D of $\mathbb{F}_{q^2}^n$ of cardinality $|D| = q^n/K$ such that $D \leq D^{\perp_a}$ and $\mathrm{wt}(D^{\perp_a} \backslash D) = d$ if $K > 1$ (and $\mathrm{wt}(D^{\perp_a}) = d$ if $K = 1$).*

According to the results above and Lemmas 6 and 7, we give the existence of quantum codes over $\mathbb{F}_q$ in the following theorem.

Theorem 2 *Let C be an $\mathbb{F}_q$-linear skew cyclic code over $\mathbb{F}_{q^2}$ with parameters $(n, M, d)_{q^2}$. Then $\mathcal{S}(C)$ is an additive skew $(2n, M, 2d)_{q^2}$-code, $\mathcal{S}(C) \subseteq \mathcal{S}(C)^{\perp_a}$ and there exists an $[[2n, k, d_Q]]_q$ quantum code, where $k = \log_q(q^{2n}/M)$, $d_Q = \mathrm{wt}(\mathcal{S}(C)^{\perp_a} \backslash \mathcal{S}(C))$ if $q^k > 1$ (and $d_Q = wt(\mathcal{S}(C)^{\perp_a})$ if $q^k = 1$).*

For the convenience and practicality of the calculation, we just consider these codes which have a single generator polynomial, that is $k(x) = 0$ in Theorem 1, in the rest of this article. Let

$$C = \langle \xi g(x) + p(x) \rangle$$

be an $\mathbb{F}_q$-linear skew cyclic code of length n over $\mathbb{F}_{q^2}$. According to the definition of the trace-alternating inner product $< \cdot, \cdot >_a$ and $|C| \cdot |C^{\perp_a}| = q^{2n}$, it is not difficult to verify that if $|C| = q^n$, then we have

$$C^{\perp_a} = \langle \xi g(x) + p(x) \rangle$$

is an $\mathbb{F}_q$-linear skew cyclic code of length n with cardinality $|C^{\perp_a}| = q^n$ over $\mathbb{F}_{q^2}$. This means that C is a trace-alternating self-dual $\mathbb{F}_q$-linear skew cyclic code over $\mathbb{F}_{q^2}$.

Let $C = C^{\perp_a} = \langle \xi g(x) + p(x) \rangle$ be a trace-alternating self-dual $\mathbb{F}_q$-linear skew cyclic code over $\mathbb{F}_{q^2}$. Then $\mathcal{S}(C) = \mathcal{S}(C^{\perp_a})$. According to Lemma 6, we have that $\mathcal{S}(C) \subseteq \mathcal{S}(C)^{\perp_a}$, which implies that $\mathcal{S}(C) = \mathcal{S}(C^{\perp_a}) \subset \mathcal{S}(C)^{\perp_a}$.

With notations and results listed above, we give some good quantum codes in the following examples.

Example 2 Let $q = 2$ and $n = 4$. Then $x^4 - 1 = (x + 1)^4$ over $\mathbb{F}_2$. Let $g(x) = (x + 1)^2 = x^2 + 1$ and $p(x) = x^3 + x^2 + x + 1$, then we have

$$C = \langle \xi_1 g(x) + p(x) \rangle = \langle x^3 + (1 + \xi_1)x^2 + x + 1 + \xi_1 \rangle$$

with generator matrix

$$G = \begin{pmatrix} 1+\xi_1 & 1 & 1+\xi_1 & 1 \\ 1 & (1+\xi_1)^2 & 1 & (1+\xi_1)^2 \end{pmatrix},$$

where ξ_1 is a primitive element of $\mathbb{F}_{2^2}$ with $\mathrm{ord}(\xi_1) = 3$. With the computational algebra system Magma [5], we have that C is an $\mathbb{F}_2$-linear skew cyclic $(4, 2^2, 4)_4$ code. According to Lemma 5, we have that $\mathcal{S}(C)$ is an additive skew $(8, 2^2, 8)_4$ 2-quasi-cyclic code with generator matrix

$$G_{S(C)} = \begin{pmatrix} 1+\xi_1 & (1+\xi_1)^2 & 1 & 1 & 1+\xi_1 & (1+\xi_1)^2 & 1 & 1 \\ 1 & 1 & (1+\xi_1)^2 & (1+\xi_1)^4 & 1 & 1 & (1+\xi_1)^2 & (1+\xi_1)^4 \end{pmatrix}.$$

Using Magma [5], we have that $\mathcal{S}(C)^{\perp_a}$ is an additive skew $(8, 2^{14}, 2)_4$ code. By Theorem 2, we obtain an optimal binary quantum code $[[8, 6, 2]]$ which has the same parameters with the best known binary additive quantum code given in [34].

Table 2 Some optimal binary quantum codes $[[n, k, d]]$

q	n	$g(x)$	$p(x)$	$[[n, k, d]]$
2	3	1	x^2+1	$[[6, 3, 2]]$
2	3	x^2+x+1	x^2+x+1	$[[6, 4, 2]]$
2	4	1	x^2+x+1	$[[8, 4, 2]]$
2	4	x^2+1	$x+1$	$[[8, 5, 2]]$
2	4	x^2+1	x^3+x^2+x+1	$[[8, 6, 2]]$
2	5	1	x^3+x+1	$[[10, 5, 2]]$
2	5	$x^4+x^3+x^2+x+1$	$x^4+x^3+x^2+x+1$	$[[10, 8, 2]]$
2	6	x^3+1	x^2+x	$[[12, 8, 2]]$

At the end of this example, we give some optimal binary quantum codes which have the same parameters with the best known binary additive quantum codes given in [34] in Table 2.

Example 3 Let $q = 3$ and $n = 3$. Then $x^3 - 1 = (x+2)^3$ over $\mathbb{F}_3$. Let $g(x) = (x+2)^2 = x^2+x+1$ and $p(x) = \xi_2^2 x + \xi_2^6$. Then

$$C = \langle \xi_2 g(x) + p(x) \rangle = \langle \xi_2 x^2 + (\xi_2 + \xi_2^2)x + \xi_2 + \xi_2^6 \rangle$$

with generator matrix

$$G = \begin{pmatrix} \xi_2+\xi_2^6 & \xi_2+\xi_2^2 & \xi_2 \\ \xi_2^3 & (\xi_2+\xi_2^6)^3 & (\xi_2+\xi_2^2)^3 \\ (\xi_2+\xi_2^2)^9 & \xi_2^9 & (\xi_2+\xi_2^6)^9 \end{pmatrix},$$

where ξ_2 is a primitive element of $\mathbb{F}_{3^2}$ with $\mathrm{ord}(\xi_2) = 8$. Using Magma [5], we have that C is an $\mathbb{F}_3$-linear skew cyclic $(3, 3^3, 2)_9$ code. By Lemma 5, we have that $\mathcal{S}(C)$ is an additive skew $(6, 3^3, 4)_9$ 2-quasi-cyclic code with generator matrix

$$G_{S(C)} = \begin{pmatrix} \xi_2+\xi_2^6 & (\xi_2+\xi_2^6)^3 & \xi_2+\xi_2^2 & (\xi_2+\xi_2^2)^3 & \xi_2 & \xi_2^3 \\ \xi_2^3 & \xi_2^9 & (\xi_2+\xi_2^6)^3 & (\xi_2+\xi_2^6)^9 & (\xi_2+\xi_2^2)^3 & (\xi_2+\xi_2^2)^9 \\ (\xi_2+\xi_2^2)^9 & (\xi_2+\xi_2^2)^{27} & \xi_2^9 & \xi_2^{27} & (\xi_2+\xi_2^6)^9 & (\xi_2+\xi_2^6)^{27} \end{pmatrix}.$$

Using Magma [5], we have that $\mathcal{S}(C)^{\perp_a}$ is an additive skew $(6, 3^9, 2)_9$ code. By Theorem 2, we obtain a new quantum code $[[6, 3, 2]]_3$ which has better parameters than the quantum code $[[6, 2, 2]]_3$ given in [2].

Let $q = 3$, $n = 4$, $g(x) = x^2 + 2$ and $p(x) = \xi_2^2 x + \xi_2^6$. According to the numerical results by using Magma [5], Lemma 5, Theorems 1 and 2, we have that $\mathcal{S}(C)$ is an additive skew $(8, 3^3, 6)_9$ 2-quasi-cyclic code and $\mathcal{S}(C)^{\perp_a}$ is an additive

skew $(8, 3^{13}, 2)_9$ code. Thus, we obtain a new quantum code $[[8, 5, 2]]_3$ which has better parameters than the quantum code $[[8, 4, 2]]_3$ given in [2].

5 Conclusion

In this paper, we study the structure of $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$. Some good $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ are constructed. Moreover, we give a method to construct quantum codes from $\mathbb{F}_q$-linear skew cyclic codes over $\mathbb{F}_{q^2}$ and some optimal and new quantum codes are obtained.

Acknowledgements This research is supported by the 973 Program of China (Grant No. 2013CB834204), the National Natural Science Foundation of China (Grant Nos. 11671024, 61571243, 11701336, 11626144 and 11671235), the Fundamental Research Funds for the Central Universities of China, the Scientific Research Fund of Hubei Provincial Key Laboratory of Applied Mathematics (Hubei University)(Grant No. HBAM201804), the Scientific Research Fund of Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science and Technology)(Grant No. 2018MMAEZD04), the Beijing Postdoctoral Research Foundation, and the Chaoyang Postdoctoral Research Foundation.

References

1. Aydin, N., Abualrub, T.: Optimal quantum codes from additive skew cyclic codes. Discret. Math., Algorithms Appl. **8**(3), 1650037 (2016)
2. Ashraf, M., Mohammad, G.: Quantum codes from cyclic codes over $\mathbb{F}_3 + v\mathbb{F}_3$. Int. J. Quantum Inf. **12**(6), 1450042 (2014)
3. Ashraf, M., Mohammad, G.: Skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$. Asian-Eur. J. Math. **11**(5), 1850072 (2018)
4. Boucher, D., Geiselmann, W., Ulmer, F.: Skew-cyclic codes. Appl. Algebra Eng. Comm. Comput. **18**, 379–389 (2007)
5. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. the user language. J. Symb. Comput. **24**, 235–265 (1997)
6. Bag, T., Ashraf, M., Mohammad, G., Upadhyay, A.K.: Quantum codes from $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic codes over the ring $\mathbb{F}_q+u_1\mathbb{F}_q+\cdots+u_{2m}\mathbb{F}_q$. Quantum Inf. Process. **18**, 270 (2019)
7. Bag, T., Upadhyay, A.K.: Skew cyclic and skew constacyclic codes over the ring $\mathbb{F}_p + u_1\mathbb{F}_p + \cdots + u_{2m}\mathbb{F}_p$. Asian-Eur. J. Math. **12**(5), 1950083 (2019)
8. Calderbank, A.R., Rains, E.M., Shor, P.M., Sloane, N.J.A.: Quantum error correction via codes over $GF(4)$. IEEE Trans. Inf. Theory **44**, 1369–1387 (1998)
9. Chen, B., Ling, S., Zhang, G.: Application of constacyclic codes to quantum MDS codes. IEEE Trans. Inf. Theory **61**(3), 1474–1484 (2015)
10. Chen, X., Zhu, S., Kai, X.: Entanglement-assisted quantum MDS codes constructed from constacyclic codes. Quantum Inf. Process. **17**(10), 273 (2018)
11. Dertli, A., Cengellenmis, Y.: Skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. J. Sci. Arts **2**(39), 215–222 (2017)
12. Diao, L., Gao, J., Lu, J.: Some results on $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes. Adv. Math. Commun. **14**(4), 555–572 (2020)
13. Dinh, H.Q., Bag, T., Upadhyay, A.K., Bandi, R., Tansuchat, R.: A class of skew cyclic codes and application in quantum codes construction. Discret. Math. **344**(2), 112189 (2021)
14. Ezerman, M.F., Ling, S., Solé, P., Yemen, O.: From skew-cyclic codes to asymmetric quantum codes. Adv. Math. Commun. **5**(1), 41–57 (2011)
15. Fang, W., Fu, F.-W.: Two new classes of quantum MDS codes. Finite Fields Appl. **53**, 85–98 (2018)
16. Gao, J., Wang, Y.: u-Constacyclic codes over $\mathbb{F}_p + u\mathbb{F}_p$ and their applications of constructing new non-binary quantum codes. Quantum Inf. Process. **17**, 4 (2018)

17. Gao, J., Ma, F., Fu, F.-W.: Skew constacyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$. Appl. Comput. Math. **16**(3), 286–295 (2017)
18. Gao, J.: Quantum codes from cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q + v^3\mathbb{F}_q$. Int. J. Quantum Inf. **13**(8), 1550063 (2015)
19. Gao, J., Wang, Y.: Quantum codes derived from negacyclic codes. Internat. J. Theoret. Phys. **57**(3), 682–686 (2018)
20. Gao, J., Wang, Y.: New non-binary quantum codes derived from a class of linear codes. IEEE Access **7**(1), 26418–26421 (2019)
21. Gao, Y., Gao, J., Fu, F.-W.: Quantum codes from cyclic codes over the ring $\mathbb{F}_q + v_1\mathbb{F}_q + \cdots + v_r\mathbb{F}_q$. Appl. Algebra Eng. Comm. Comput. **30**(2), 161–174 (2019)
22. Gao, Y., Yang, S., Fu, F.-W.: Some optimal cyclic $\mathbb{F}_q$-linear $\mathbb{F}_{q^t}$-codes. Adv. Math. Commun. (2020). https://doi.org/10.3934/amc.2020072
23. Galindo, C., Hernando, F., Matsumoto, R.: Quasi-cyclic constructions of quantum codes. Finite Fields Appl. **52**, 261–280 (2018)
24. Gluesing-Luerssen, H., Pllaha, T.: On quantum stabilizer codes derived from local frobenius rings. Finite Fields Appl. **58**, 145–173 (2019)
25. Gursoy, F., Siap, I., Yildiz, B.: Construction of skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$. Adv. Math. Commun. **8**, 313–322 (2014)
26. Jin, L.: Quantum stabilizer codes from maximal curves. IEEE Trans. Inf. Theory **60**(1), 313–316 (2014)
27. Ketkar, A., Klappenecker, A., Kumar, S., Sarvepalli, P.K.: Nonbinary stabilizer codes over finite fields. IEEE Trans. Inf. Theory **52**(11), 4892–4914 (2006)
28. Kai, X., Zhu, S.: New quantum MDS codes from negacyclic codes. IEEE Trans. Inf. Theory **59**(2), 1193–1197 (2013)
29. Lidl, R., Niederreiter, H., Cohn, P.M.: Finite fields. Cambridge University Press, Cambridge (1997)
30. Luo, G., Cao, X.: Two new families of entanglement-assisted quantum MDS codes from generalized Reed-Solomon codes. Quantum Inf. Process. **18**(3), 89 (2019)
31. Li, R., Wang, J., Liu, Y., Guo, G.: New quantum constacyclic codes. Quantum Inf. Process. **18**, 127 (2019)
32. Liu, X., Liu, H.: Quantum codes from linear codes over finite chain rings. Quantum Inf. Process. **16**(10), 240 (2017)
33. Liu, X., Yu, L., Hu, P.: New entanglement-assisted quantum codes from k-Galois dual codes. Finite Fields Appl. **55**, 21–32 (2019)
34. Markus, G.: Bounds on the minimum distance of linear codes and quantum codes. Online available at http://www.codetables.de. Accessed on 24 Oct 2019
35. Ma, F., Gao, J., Fu, F.-W.: Constacyclic codes over the ring $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q$ and their applications of constructing new non-binary quantum codes. Quantum Inf. Process. **17**(6), 122 (2018)
36. Ma, F., Gao, J., Fu, F.-W.: New non-binary quantum codes from constacyclic codes over $\mathbb{F}_q[u, v]/<u^2 - 1, v^2 - v, uv - vu>$. Adv. Math. Commun. **13**(3), 421–434 (2019)
37. Siap, I., Abualrub, T., Aydin, N., Seneviratne, P.: Skew cyclic codes of arbitrary length. Int. J. Inf. Codin. Theory **2**, 10–20 (2011)
38. Shi, X., Yue, Q., Zhu, X.: Construction of some new quantum MDS codes. Finite Fields Appl. **46**, 347–362 (2017)
39. Shor, P.W.: Scheme for reducing decoherence in quantum memory. Phys. Rev. A **52**, 2493–2496 (1995)
40. Zhang, T., Ge, G.: Quantum MDS codes with large minimum distance. Des. Codes Cryptogr. **83**(3), 503–517 (2017)